

Generalized Lüscher's Formula in Multichannel Baryon-Baryon Scattering

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In this paper, Lüscher's formula is generalized to the case of two spin- $\frac{1}{2}$ particles in two-channel scattering based on Ref. [1]. This is first done in a non-relativistic quantum mechanics model and then generalized to quantum field theory. We show that Lüscher's formula obtained from these two different methods are equivalent up to terms that are exponentially suppressed in the box size. This formalism can be readily applied to future lattice QCD calculations.

I. INTRODUCTION

The scattering of two baryons is important for the study of strong interaction which is one of the four elementary interactions in Nature. The interactions among baryons are also relevant for the understanding of nuclear matter which is also crucial in other fields of physics. However, due to its non-perturbative nature at low-energies, theoretical study of baryon interactions requires non-perturbation methods such as lattice chromodynamics (lattice QCD). Lattice QCD is a non-perturbative method implemented in discretized Euclidean space-time. Within this formalism, physical quantities are encoded into various Euclidean correlation functions which in turn can be measured using Monte Carlo simulations. Since the lattice calculations are all performed in a finite volume, the quantities obtained in lattice simulations need to be transformed into physical quantities that are measured in experiments. For the study of hadron-hadron interactions at low energies, Lüscher has set up a formalism which relates the two-particle elastic scattering S -matrix elements with the corresponding two-particle energies in a finite volume [2–5]. Since the advent of Lüscher's formula, various lattice studies, both quenched [6–13] and unquenched [14–23], have been performed over the years to investigate the scattering of hadrons.

The original Lüscher's formula was derived for two spinless massive particles in the center of mass (COM) frame below the inelastic threshold. It has been generalized in various ways over the years: to the boosted frames [24–26], to the case of asymmetric boxes [27–29], with twisted boundary conditions [30–33], to the case of particles with spin, e.g. to the case of two baryons [29, 34–40] and beyond the inelastic threshold [41–46], and even to three-particle case [47–50].

In our previous study [1], we have generalized the formalism to the case of multi-channel scattering of a spinless particle with a spin- $\frac{1}{2}$ particle in arbitrary frame. In this paper, we continue to synthesize Lüscher's formula for the multi-channel two-particle scattering with each of the two hadrons having spin $\frac{1}{2}$. We will call them

baryons for simplicity and they in principle can be of the same type (e.g. two protons) or different (e.g. one proton and one neutron). Beyond the inelastic threshold, the scattering becomes multi-channel. In this paper, we only study the two-channel scattering case. It can be generalized readily to other multichannel scattering cases easily. In a recent comprehensive study, Briceno has studied the most general two-particle scenario [51].

The organization of this paper is as follows: In Sec. II, we start the discussion in the case of non-relativistic quantum mechanics. In Sec. III, generalization to quantum field theory is done. We also compare the results obtained within quantum field theory with those obtained in previous section and show that they are actually equivalent up to terms that are exponentially suppressed in the box size. In Sec. IV, we will give the conclusions. Possible applications of these formulae in future lattice computations are also discussed. Some explicit formulas used in this paper are listed in the appendix for reference.

II. LÜSCHER'S FORMULA FOR BARYON-BARYON SCATTERING IN NON-RELATIVISTIC QUANTUM MECHANICS

We start our discussion from a non-relativistic quantum mechanics model in infinite volume in the continuum. The model we use here is the same as that of Lüscher [4] in which a finite-ranged potential is assumed. After singling out the center of mass motion, one focuses on the relative motion of the two baryons. In the center-of-mass (COM) frame, the energy is given by

$$E = \frac{\mathbf{k}_1^2}{2\mu_1} = E_T + \frac{\mathbf{k}_2^2}{2\mu_2} \quad (1)$$

where μ_1 and μ_2 is the reduced mass of the two-particle system below and above the threshold E_T , respectively. Since the potential has a finite range, in the large r region where the potential vanishes, the wave function of the scattering states can be written as

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$$\psi_{1;s\nu}(\mathbf{r}) \xrightarrow{r \rightarrow \infty} \left(\chi_{s\nu} e^{i\mathbf{k}_1 \cdot \mathbf{r}} + \sum_{s'\nu'} \chi_{s'\nu'} M_{11;s'\nu';s\nu}^{(NR)} \frac{e^{ik_1 r}}{r} \right). \quad (2)$$

This wavefunction has the property that, in the infinite past, it reduces to an incident plane wave in the first channel. The symbol $\chi_{s\nu}$ designates spin-wavefunction which is an eigenstate of spin angular momentum of S^2 and s_z with the eigenvalues given by $s = 0, \nu = 0$ (singlet state), or $s = 1, \nu = 1, -1, 0$ (triplet state). $M_{if;s'\nu';s\nu}^{(NR)}$ is the scattering amplitude. We have added a superscript (NR) to distinguish the scattering amplitude introduced here (non-relativistic quantum mechanics) with that introduced in quantum field theory later on. Subscripts

like i and f correspond to the channels and take the value 1 or 2 in this paper. There is another analogous but linearly independent wavefunction given by

$$\psi_{2;s\nu}(\mathbf{r}) \xrightarrow{r \rightarrow \infty} \left(\sqrt{\frac{\mu_1}{\mu_2}} \sum_{s'\nu'} \chi_{s'\nu'} M_{12;s'\nu';s\nu}^{(NR)} \frac{e^{ik_1 r}}{r} + \chi_{s\nu} e^{i\mathbf{k}_2 \cdot \mathbf{r}} + \sum_{s'\nu'} \chi_{s'\nu'} M_{22;s'\nu';s\nu}^{(NR)} \frac{e^{ik_2 r}}{r} \right) \quad (3)$$

which reduces to an incident plane wave in the second channel. Choosing the z-axis to coincide with either \mathbf{k}_1 or \mathbf{k}_2 , the scattering amplitudes introduced above are related to the S -matrix elements as [52]

$$M_{11;s'\nu';s\nu}^{(NR)}(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{r}}) = \frac{1}{2ik_1} \sum_{l'=0}^{\infty} \sum_{l=0}^{\infty} \sum_{J=l-1}^{l+1} \sqrt{4\pi(2l+1)} i^{(l-l')} (S_{11;l's';ls}^J - \delta_{ll'} \delta_{ss'}) \langle JM|l'm';s'\nu' \rangle \langle JM|l0;s\nu \rangle Y_{l'm'}(\hat{\mathbf{r}}) \quad (4)$$

$$M_{12;s'\nu';s\nu}^{(NR)}(\hat{\mathbf{k}}_2 \cdot \hat{\mathbf{r}}) = \frac{1}{2i\sqrt{k_1 k_2}} \sum_{l'=0}^{\infty} \sum_{l=0}^{\infty} \sum_{J=l-1}^{l+1} \sqrt{4\pi(2l+1)} i^{(l-l')} S_{12;l's';ls}^J \langle JM|l'm';s'\nu' \rangle \langle JM|l0;s\nu \rangle Y_{l'm'}(\hat{\mathbf{r}}). \quad (5)$$

Comparing with Eq. (2), Eq. (3), Eq. (4) and Eq. (5), we obtain the following form as:

$$\psi_{i;s\nu}(\mathbf{r}) = \sum_{s'JMl'} \sqrt{4\pi(2l+1)} W_{i;l's';ls}^J \langle JM|l0;s\nu \rangle Y_{JM}^{l's'}(\hat{\mathbf{r}}) \quad (6)$$

where $Y_{JM}^{l's'}(\hat{\mathbf{r}})$ is the spin spherical harmonics whose explicit form is given by

$$Y_{JM}^{ls}(\hat{\mathbf{r}}) = \sum_{m\nu} Y_{lm}(\hat{\mathbf{r}}) \chi_{s\nu} \langle JM|lm;s\nu \rangle. \quad (7)$$

In Eq. (6), $W_{i;l's';ls}^J(r)$ is the radial wave function of the two-particle scattering states. In the large r region, they have the following asymptotic forms:

$$W_{1;l's';ls}^J(r) = \left(\frac{1}{2ik_1 r} i^{(l-l')} [S_{11;l's';ls}^J e^{ik_1 r} + (-1)^{l+1} e^{-ik_1 r} \delta_{ll'} \delta_{ss'}] + \frac{1}{2ir\sqrt{k_1 k_2}} \sqrt{\frac{\mu_2}{\mu_1}} i^{(l-l')} S_{21;l's';ls}^J e^{ik_2 r} \right), \quad (8)$$

$$W_{2;l's';ls}^J(r) = \left(\frac{1}{2ir\sqrt{k_1 k_2}} \sqrt{\frac{\mu_1}{\mu_2}} i^{(l-l')} S_{12;l's';ls}^J e^{ik_1 r} + \frac{1}{2ik_2 r} i^{(l-l')} [S_{22;l's';ls}^J e^{ik_2 r} + (-1)^{l+1} e^{-ik_2 r} \delta_{ll'} \delta_{ss'}] \right). \quad (9)$$

Now we enclose the two-particle system in a cubic box of size L and impose periodic boundary condition. In the outer region where the potential vanishes, the wave function becomes

$$\psi(\mathbf{r}) = \sum_{s'JMl's} \left[\sum_{i=1}^2 F_{i;JMls} W_{i;l's';ls}^J(r) \right] Y_{JM}^{l's'}(\hat{\mathbf{r}}). \quad (10)$$

On the other hand, also in this region, the wave function is a linear superposition of the singular periodic solutions, $G_{i;JMls}(\mathbf{r}; k_i^2)$, of the Helmholtz equation. Thus, we also

get

$$\psi(\mathbf{r}) = \left(\frac{\sum_{s=0}^1 \sum_{l=0}^{\infty} \sum_{J=l-\frac{1}{2}}^{l+1} \sum_{M=-J}^J v_{1;JMs} G_{1;JMs}(\mathbf{r}; k_1^2)}{\sum_{s=0}^1 \sum_{l=0}^{\infty} \sum_{J=l-\frac{1}{2}}^{l+\frac{1}{2}} \sum_{M=-J}^J v_{2;JMs} G_{2;JMs}(\mathbf{r}; k_2^2)} \right) \quad (11)$$

$$G_{i;JMs} = \frac{(-1)^l k_i^{l+1}}{4\pi} (Y_{JM}^{ls} n_l(k_i r) + \sum_{J'M'l'} \mathcal{M}_{i;JMs;J'M'l'}^{(s)}(k_i^2) Y_{J'M'l'}^{l's}(k_i r)) , \quad (12)$$

The explicit form of $\mathcal{M}_{i;JMs;J'M'l'}^{(s)}(k_i^2)$ can be found, for example, in Ref. [1]. Comparison of Eq. (10) with Eq. (11) then leads to four linear equations of the coef-

The singular solutions of the Helmholtz equation $G_{i;JMs}(\mathbf{r}; k_i^2)$ can be expanded in terms of spherical harmonics as,

ficients. In order to have non-trivial solutions for them, the determinant of the corresponding matrix must vanish which leads to the basic form of Lüscher's formula:

$$\begin{vmatrix} \sum_{l''} (S_{11;l''s';ls}^J - \delta_{ll''} \delta_{ss'}) \mathcal{M}_{1;JMs;J'M'l'}^{(s')} - i \delta_{JJ'} \delta_{MM'} (S_{11;l's';ls}^J + \delta_{ll'} \delta_{ss'}) & \sqrt{\frac{k_2}{k_1}} (\sum_{l''} S_{21;l''s';ls}^J \mathcal{M}_{2;JMs;J'M'l'}^{(s')} - i S_{21;l's';ls}^J \delta_{JJ'} \delta_{MM'}) \\ \sqrt{\frac{k_1}{k_2}} (\sum_{l''} S_{12;l''s';ls}^J \mathcal{M}_{1;JMs;J'M'l'}^{(s')} - i S_{12;l's';ls}^J \delta_{JJ'} \delta_{MM'}) & \sum_{l''} (S_{22;l''s';ls}^J - \delta_{ll''} \delta_{ss'}) \mathcal{M}_{2;JMs;J'M'l'}^{(s')} - i \delta_{JJ'} \delta_{MM'} (S_{22;l's';ls}^J + \delta_{ll'} \delta_{ss'}) \end{vmatrix} = 0 . \quad (13)$$

In a definite irreducible representation (irrep) of the cubic group, the basis vectors are labelled as $|\Gamma, \xi, J, l, s, n\rangle$ where Γ denotes the representation; ξ runs from 1 to the number of the dimension and n runs from 1 to the multiplicity of the representation. This basis can

be expressed by linear combinations of $|JMs\rangle$ and the corresponding matrix \mathcal{M}_i is diagonal with respect to Γ and ξ by Schur's lemma [4]. Therefore, in a particular symmetry sector Γ , Lüscher's formula becomes

$$\begin{vmatrix} \sum_{l''} (S_{11;l''s';ls}^J - \delta_{ll''} \delta_{ss'}) \mathcal{M}_{1;Jl'';J'l'}^{(s')}(\Gamma) - i \delta_{JJ'} (S_{11;l's';ls}^J + \delta_{ll'} \delta_{ss'}) & \sqrt{\frac{k_2}{k_1}} (\sum_{l''} S_{21;l''s';ls}^J \mathcal{M}_{2;Jl'';J'l'}^{(s')}(\Gamma) - i S_{21;l's';ls}^J \delta_{JJ'}) \\ \sqrt{\frac{k_1}{k_2}} (\sum_{l''} S_{12;l''s';ls}^J \mathcal{M}_{1;Jl'';J'l'}^{(s')}(\Gamma) - i S_{12;l's';ls}^J \delta_{JJ'}) & \sum_{l''} (S_{22;l''s';ls}^J - \delta_{ll''} \delta_{ss'}) \mathcal{M}_{2;Jl'';J'l'}^{(s')}(\Gamma) - i \delta_{JJ'} (S_{22;l's';ls}^J + \delta_{ll'} \delta_{ss'}) \end{vmatrix} = 0 . \quad (14)$$

For the case of integral total angular momentum J , we need to consider the cubic group O which has five irrep's: A_1 , A_2 , E , T_1 and T_2 with dimensions 1, 1, 2, 3 and 3.

For two spin $\frac{1}{2}$ particles, the total spin quantum numbers of the system can take 0 (singlet state) or 1 (triplet states). The parity of the states depends only on orbital angular momentum quantum number and is given by $(-)^l$. Thus, for the singlet state and the triplet states

with $l = J$, parity is simply $(-)^J$ while for the other cases it is $(-)^{J+1}$. For parity-conserving theories like QCD, there is no scattering between states with opposite parity[52]. Then we should divide the Lüscher's formula into the case a and case b, they corresponding to the states of parity $(-1)^{J+1}$ and parity $(-1)^J$ respectively.

case a. $s = s' = 1$, $l = J \pm 1$

Lüscher's formula becomes

$$\left| \begin{array}{cc} \sum_{l''} (S_{11;l''1;l1}^J - \delta_{ll''}) \mathcal{M}_{1;Jl'';J'l'}^{(1)} - i\delta_{JJ'} (S_{11;l'1;l1}^J + \delta_{ll'}) & \sqrt{\frac{k_2}{k_1}} (\sum_{l''} S_{21;l''1;l1}^J \mathcal{M}_{2;Jl'';J'l'}^{(1)} - iS_{21;l'1;l1}^J \delta_{JJ'}) \\ \sqrt{\frac{k_1}{k_2}} (\sum_{l''} S_{12;l''1;l1}^J \mathcal{M}_{1;Jl'';J'l'}^{(1)} - iS_{12;l'1;l1}^J \delta_{JJ'}) & \sum_{l''} (S_{22;l''1;l1}^J - \delta_{ll''}) \mathcal{M}_{2;Jl'';J'l'}^{(1)} - i\delta_{JJ'} (S_{22;l'1;l1}^J + \delta_{ll'}) \end{array} \right| = 0. \quad (15)$$

If we consider the explicit parity and assume that the cutoff angular momentum is $\Lambda = 4$, the the decomposition in this case is as follows: $0^- = A_1^-$, $1^+ = T_1^+$, $2^- = E^- + T_2^-$, $3^+ = A_2^+ + T_1^+ + T_2^+$, and $4^- =$

$A_1^- + E^- + T_1^- + T_2^-$. In the following, we will only focus on A_1^- representation. Neglecting the mixing between $J = 0$ and $J = 4$, Lüscher's formula is given by

$$\left| \begin{array}{cc} (S_{11;11;11}^0 - 1) \mathcal{M}_{1;01;01}^{(1)} - i(S_{11;11;11}^0 + 1) & \sqrt{\frac{k_2}{k_1}} (S_{21;11;11}^0 \mathcal{M}_{2;01;01}^{(1)} - iS_{21;11;11}^0) \\ \sqrt{\frac{k_1}{k_2}} (S_{12;11;11}^0 \mathcal{M}_{1;01;01}^{(1)} - iS_{12;11;11}^0) & (S_{22;11;11}^0 - 1) \mathcal{M}_{2;01;01}^{(1)} - i(S_{22;11;11}^0 + 1) \end{array} \right| = 0. \quad (16)$$

case b: $l = l' = J$, $s = 0, 1$.

For this case, the Lüscher's formula can be expressed as:

$$\left| \begin{array}{cc} \sum_{l''} (S_{11;l''s';ls}^J - \delta_{ll''} \delta_{ss'}) \mathcal{M}_{1;Jl'';J'l'}^{(s')} - i\delta_{JJ'} (S_{11;ls';ls}^J + \delta_{ss'}) & \sqrt{\frac{k_2}{k_1}} (\sum_{l''} S_{21;l''s';ls}^J \mathcal{M}_{2;Jl'';J'l'}^{(s')} - iS_{21;ls';ls}^J \delta_{JJ'}) \\ \sqrt{\frac{k_1}{k_2}} (\sum_{l''} S_{12;l''s';ls}^J \mathcal{M}_{1;Jl'';J'l'}^{(s')} - iS_{12;ls';ls}^J \delta_{JJ'}) & \sum_{l''} (S_{22;l''s';ls}^J - \delta_{ll''} \delta_{ss'}) \mathcal{M}_{2;Jl'';J'l'}^{(s')} - i\delta_{JJ'} (S_{22;ls';ls}^J + \delta_{ss'}) \end{array} \right| = 0. \quad (17)$$

If we consider the explicit parity and also suppose that the cutoff angular momentum is $\Lambda = 4$, the decomposition in this case becomes $0^+ = A_1^+$, $1^- = T_1^-$, $2^+ = E^+ + T_2^+$, $3^- = A_2^- + T_1^- + T_2^-$ and $4^+ = A_1^+ + E^+ + T_1^+ + T_2^+$. Focusing on the A_1^+ representation which corresponds $J = 0$ and ignoring the index l and l' , both of which are unity, a similar formula can be easily obtained from Eq. (17).

The above discussion has not taken into account the possibility for the identical nature of the two particles in which case singlet-triplet transition within the same parity is allowed. However, if we further assume that the two particles are identical, then singlet-triplet transition is forbidden since the singlet state has an antisymmetric spin wave function which then requires a symmetric spatial one that necessarily has *positive* parity while the triplet states have the opposite parity. Below, we list Lüscher formulae for $s = s' = 0$ and $s = s' = 1$ cases respectively.

$$\left| \begin{array}{cc} \sum_{l''} (S_{11;l''0;l0}^J - \delta_{ll''}) \mathcal{M}_{1;Jl'';J'l'}^{(0)} - i\delta_{JJ'} (S_{11;l0;l0}^J + 1) & \sqrt{\frac{k_2}{k_1}} (\sum_{l''} S_{21;l''0;l0}^J \mathcal{M}_{2;Jl'';J'l'}^{(0)} - iS_{21;l0;l0}^J \delta_{JJ'}) \\ \sqrt{\frac{k_1}{k_2}} (\sum_{l''} S_{12;l''0;l0}^J \mathcal{M}_{1;Jl'';J'l'}^{(0)} - iS_{12;l0;l0}^J \delta_{JJ'}) & \sum_{l''} (S_{22;l''0;l0}^J - \delta_{ll''}) \mathcal{M}_{2;Jl'';J'l'}^{(0)} - i\delta_{JJ'} (S_{22;l0;l0}^J + 1) \end{array} \right| = 0. \quad (18)$$

$$\begin{vmatrix}
\sum_{l''} (S_{11;l''1;l1}^J - \delta_{ll''}) \mathcal{M}_{1;Jl'';J'l'}^{(1)} - i\delta_{JJ'} (S_{11;l1;l1}^J + 1) & \sqrt{\frac{k_2}{k_1}} (\sum_{l''} S_{21;l''1;l1}^J \mathcal{M}_{2;Jl'';J'l'}^{(1)} - iS_{21;l1;l1}^J \delta_{JJ'}) \\
\sqrt{\frac{k_1}{k_2}} (\sum_{l''} S_{12;l''1;l1}^J \mathcal{M}_{1;Jl'';J'l'}^{(1)} - iS_{12;l1;l1}^J \delta_{JJ'}) & \sum_{l''} (S_{22;l''1;l1}^J - \delta_{ll''}) \mathcal{M}_{2;Jl'';J'l'}^{(1)} - i\delta_{JJ'} (S_{22;l1;l1}^J + 1)
\end{vmatrix} = 0. \quad (19)$$

From these explicit expressions, it is seen that they are quite similar as those in the case of meson-meson two-channel scattering [42].

Finally, let us comment briefly on Lüscher formula in moving frames (MF). In this case, one should expanded the wave function of the system in the outer region in terms of modified Green's function $G_{i;JMLs}^{\mathbf{d}}(\mathbf{r}; k_i^{*2})$ and modified matrix $\mathcal{M}_{i;JML;J'M'l'}^{(s)}(k_i^{*2})$ [24]. The explicit forms of these quantities can be obtained by substituting $Y_{JM}^{l\frac{1}{2}}$ for Y_{JM}^{ls} , and $\langle JM|lm; \frac{1}{2}\nu \rangle$ for $\langle JM|lm; s\nu \rangle$ in corresponding formulae in Ref. [1]. Lüscher's formula takes exactly the same form as Eq.(15) and (17) except that all the labels of $\mathcal{M}_{i;JML;J'M'l'}^{(s)}(k_i^{*2})$ are replaced by $\mathcal{M}_{i;JML;J'M'l'}^{\mathbf{d}(s)}(k_i^{*2})$. Apart from the above mentioned substitutions, extra attention should also be paid to the difference in symmetry. In order to discuss Lüscher's formula in MF, we should introduce the space group \mathcal{G} , which is a semi-direct product of lattice translational group \mathcal{T} and the cubic group O . The representations are characterized by the little group Γ and the corresponding total momentum \mathbf{P} . For example, for the cases $\mathbf{P} = \frac{2\pi}{L}\mathbf{e}_3$, $\mathbf{P} = \frac{2\pi}{L}(\mathbf{e}_1 + \mathbf{e}_2)$, and $\mathbf{P} = \frac{2\pi}{L}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$, the corresponding little groups are C_{4v} , C_{2v} and C_{3v} , respectively [26]. Then, following similar steps as in the COM frame, one can easily obtain the explicit formulas for these different little groups.

III. LÜSCHER'S FORMULA FOR BARYON-BARYON SCATTERING IN QUANTUM FIELD THEORY

In this section, we will discuss the same problem in quantum field theory. Lüscher's formula will be obtained in the case of single-channel and then generalized to the case of multi-channel, following similar ideas in [1, 25, 44]. The procedure is similar to that in Refs. [1, 25, 44] so we will be quite brief here and the reader is directed to those references for further details.

It is well-known that two-particle spectrum of the system can be determined from appropriate correlation functions:

$$C(P) = \int_{L;x} e^{i(-\mathbf{P}\cdot\mathbf{x} + Ex^0)} \langle 0 | \sigma(x) \sigma^\dagger(0) | 0 \rangle, \quad (20)$$

where $P = (E, \mathbf{P})$ is the total four-momentum of the two-particle system. The interpolating operator $\sigma(x)$ is

chosen to have an overlap with the two-particle states in question and $\int_{L;x} = \int_L d^4x$ stands for the space-time integration over the finite volume. Two-particle spectra correspond to poles of $C(P)$ in the E plane. The correlation function $C(P)$ may be expressed in terms of Bethe-Salpeter amplitude iK [1, 25, 44]:

$$\begin{aligned}
C(P) &= \int_{L;q} \sigma_q [Z_1 \Delta_1 \otimes Z_2 \Delta_2]_q \sigma_q^\dagger \\
&+ \int_{L;q,q'} \sigma_q [Z_1 \Delta_1 \otimes Z_2 \Delta_2]_q iK_{q,q'} [Z_1 \Delta_1 \otimes Z_2 \Delta_2]_{q'} \sigma_{q'}^\dagger + \dots
\end{aligned} \quad (21)$$

where we have adopted the notation for the two-particle propagators:

$$Z_n(q) \Delta_n(q) = \int d^4x e^{iqx} \langle \psi_n(x) \bar{\psi}_n(0) \rangle \quad (22)$$

with $n = 1, 2$ denoting two particles whose propagators are given by

$$\Delta_n(q) = \frac{i(q^\mu \gamma_\mu + m_n)}{q^2 - m_n^2 + i\epsilon} \quad (23)$$

where m_1 and m_2 are the mass of the two baryons. The notation $Z_1 \Delta_1 \otimes Z_2 \Delta_2$ in Eq. (21) denotes a direct product in Dirac space since each of the propagators is a spinor in this space. Following similar steps as in Refs. [1, 25, 44], the correlation function can be separated into two parts: $C(P) = C^\infty(P) + C^{FV}(P)$, with the $C^\infty(P)$ being the infinite-volume limit contribution while the second part $C^{FV}(P)$ being the finite-volume corrections that contains the finite-volume two-particle poles, which is what we are interested in. The explicit expression of C^{FV} may be written as

$$\begin{aligned}
C^{FV} &= -A'FA + A'F(iM)FA + \dots \\
&= -A'F \frac{1}{1 + iMF} A
\end{aligned} \quad (24)$$

where A/A' is the Bethe-Salpeter amplitude for the initial/final state, F represents the factor associated with the two-particle loop integration/summation and M is the scattering amplitude. In the COM frame, we denote

$$F = C(\mathbf{q}^*) \tilde{F}, \quad M = C(\mathbf{q}^*)^{-1} \tilde{M}. \quad (25)$$

where the definition of $C(\mathbf{q}^*)$ can be found in the appendix c.f. Eq. (A6). Basically, the interpolating op-

erator σ_q is the product of two interpolating operators for the two hadrons and without loss of generality, we could assume each of the two particles has definite parity. Then, we insert parity projection operator $\mathcal{P} = \mathcal{P}_{1\pm} \otimes \mathcal{P}_{2\pm}$ with $\mathcal{P}_{1\pm} = \frac{1 \pm \gamma_0}{2}$ and $\mathcal{P}_{2\pm} = \frac{1 \pm \gamma_0}{2}$ in the correlation function (21). Thus, $C(\mathbf{q}^*)$ can be viewed as a 4×4 matrix, and Eq. (24) becomes

$$\begin{aligned} C^{FV} &= -AFA' + AF(iM)FA' + \dots \\ &= -AC(\mathbf{q}^*)\tilde{F}A' + AC(\mathbf{q}^*)\tilde{F}(i\tilde{M})\tilde{F}A' + \dots \\ &= -AC(\mathbf{q}^*)\tilde{F}\frac{1}{1+i\tilde{M}\tilde{F}}A'. \end{aligned} \quad (26)$$

Lüscher's formula can be obtained by requiring the finite correlator in Eq. (26) to have divergent eigenvalues. Thus the so-called quantization condition is

$$\det(1 + i\tilde{M}\tilde{F}) = 0. \quad (27)$$

In order to compare the formulae obtained in quantum field theory and those in non-relativistic quantum mechanics, we also need the relation between the two versions of the scattering matrix \tilde{M} and M^{NR} . This has been obtained in Ref. [1] which we quote here:

$$\tilde{M}_{JMs;J'M'l's'} = 8\pi E^* M_{JMs;J'M'l's'}^{(NR)}. \quad (28)$$

The quantity $M_{JMs;J'M'l's'}^{NR}$ is further related to the S -matrix elements via

$$M_{JMs;J'M'l's'}^{(NR)} = \delta_{JJ'}\delta_{MM'}\frac{1}{2ik}i^{l-l'}(S_{l's';ls}^J - \delta_{ll'}\delta_{ss'}) \quad (29)$$

As we have discussed, the two baryon system can form a singlet or a triplet state in terms of spin. For parity-conserving theories, these operators are categorized into two cases:

case a. $s = s' = 1$, $l = J \pm 1$.

In this case, we can expand the amplitude A/A' , loop factor \tilde{F} and the scattering amplitude \tilde{M} in terms of spin spherical functions:

$$\begin{cases} A(\hat{\mathbf{k}}^*) = \sqrt{4\pi}A_{JMl1}Y_{JM}^{l1\dagger}(\hat{\mathbf{k}}^*) \\ A'(\hat{\mathbf{k}}^*) = \sqrt{4\pi}A'_{JMl1}Y_{JM}^{l1}(\hat{\mathbf{k}}^*) \end{cases}, \quad (30)$$

and

$$\begin{cases} \tilde{F}(\hat{\mathbf{k}}^*, \hat{\mathbf{k}}'^*) = \frac{-1}{4\pi}\tilde{F}_{JMl1;J'M'l'1}Y_{JM}^{l1}(\hat{\mathbf{k}}^*)Y_{J'M'}^{l'1\dagger}(\hat{\mathbf{k}}'^*) \\ \tilde{M}(\hat{\mathbf{k}}^*, \hat{\mathbf{k}}'^*) = 4\pi\tilde{M}_{JMl1;J'M'l'1}Y_{JM}^{l1}(\hat{\mathbf{k}}^*)Y_{J'M'}^{l'1\dagger}(\hat{\mathbf{k}}'^*) \end{cases} \quad (31)$$

Note that, in the above expressions, $Y_{JM}^{l1}(\hat{\mathbf{k}}^*)$ only has two components with $s = 1$ and $\nu = \pm 1$. When substituting these expansions into the general formula (27), one obtains

$$\det\left(\sum_{l''} i^{(l'-l'')}(S_{l''1;l1}^J - \delta_{ll''})F_{1;Jl'';J'l'}^{FV(1)} - i\delta_{JJ'}(S_{l'1;l1}^J + \delta_{ll'})\right) = 0 \quad (32)$$

So far, the discussion has been in the case of single channel scattering. Generalization to the multi-channel case is straightforward [1, 44]. Take the two-channel case

as an example, the amplitudes A and A' in Eq. (30) become two-component vectors while both \tilde{F} and \tilde{M} in Eq. (31) become 2×2 matrices in the so-called “channel space”. Lüscher's formula then becomes

$$\left| \begin{array}{cc} \sum_{l''} i^{(l'-l'')}(S_{11;l''1;l1}^J - \delta_{ll''})F_{1;Jl'';J'l'}^{FV(1)} & \sqrt{\frac{k_2}{k_1}}(\sum_{l''} i^{(l'-l'')}S_{21;l''1;l1}^J F_{2;Jl'';J'l'}^{FV(1)} - iS_{21;l'1;l1}^J \delta_{JJ'}) \\ -i\delta_{JJ'}(S_{11;l'1;l1}^J + \delta_{ll'}) & \sum_{l''} i^{(l'-l'')}(S_{22;l''1;l1}^J - \delta_{ll''})F_{2;Jl'';J'l'}^{FV(1)} \\ \sqrt{\frac{k_1}{k_2}}(\sum_{l''} i^{(l'-l'')}S_{12;l''1;l1}^J F_{1;Jl'';J'l'}^{FV(1)} - iS_{12;l'1;l1}^J \delta_{JJ'}) & -i\delta_{JJ'}(S_{22;l'1;l1}^J + \delta_{ll'}) \end{array} \right| = 0. \quad (33)$$

case b. $l = l' = J$, $s = 0, 1$.

read:

In this case, we could also expand the amplitudes, loop factor and the scattering amplitude in terms of spin spherical functions. The single-channel case formula

$$\begin{cases} A(\hat{\mathbf{k}}^*) = \sqrt{4\pi}A_{JMls}Y_{JM}^{ls\dagger}(\hat{\mathbf{k}}^*) \\ A'(\hat{\mathbf{k}}^*) = \sqrt{4\pi}A'_{JMls}Y_{JM}^{ls}(\hat{\mathbf{k}}^*) \end{cases}, \quad (34)$$

and

$$\begin{cases} \tilde{F}(\hat{\mathbf{k}}^*, \hat{\mathbf{k}}^{*\prime}) = \frac{-1}{4\pi} \tilde{F}_{J M l s; J' M' l' s'} Y_{J M}^{l s}(\hat{\mathbf{k}}^*) Y_{J' M'}^{l' s' \dagger}(\hat{\mathbf{k}}^{*\prime}) \\ \tilde{M}(\hat{\mathbf{k}}^*, \hat{\mathbf{k}}^{*\prime}) = 4\pi \tilde{M}_{J M l s; J' M' l' s'} Y_{J M}^{l s}(\hat{\mathbf{k}}^*) Y_{J' M'}^{l' s' \dagger}(\hat{\mathbf{k}}^{*\prime}) \end{cases} \quad (35)$$

Here $Y_{J M}^{l s}(\hat{\mathbf{k}}^*)$ only have two components with $s = 1, \nu = 0$, and $s = 0, \nu = 0$, and $J = l, J' = l'$. When substituting these explicit expressions of \tilde{M} and \tilde{F} into the general formula (27) and follow similar steps as in *case a*, we can obtain the two-channel scattering Lüscher's formula as

$$\begin{vmatrix} \sum_{l''} i^{(l'-l'')} (S_{11;l''s';ls}^J - \delta_{ll''} \delta_{ss'}) F_{1;Jl'';J'l'}^{FV(s')} \\ -i\delta_{JJ'} (S_{11;ls';ls}^J + \delta_{ss'}) \\ \sqrt{\frac{k_1}{k_2}} (\sum_{l''} i^{(l'-l'')} S_{12;l''s';ls}^J F_{1;Jl'';J'l'}^{FV(s')} \\ -iS_{12;ls';ls}^J \delta_{JJ'}) \\ \sqrt{\frac{k_2}{k_1}} (\sum_{l''} i^{(l'-l'')} S_{21;l''s';ls}^J F_{2;Jl'';J'l'}^{FV(s')} \\ -iS_{21;ls';ls}^J \delta_{JJ'}) \\ \sum_{l''} i^{(l'-l'')} (S_{22;l''s';ls}^J - \delta_{ll''} \delta_{ss'}) F_{2;Jl'';J'l'}^{FV(s')} \\ -i\delta_{JJ'} (S_{22;ls';ls}^J + \delta_{ss'}) \end{vmatrix} = 0 \quad (36)$$

In this case, if the two particles are identical, there is no transition between the $s = 0, \nu = 0$ and $s = 1, \nu = 0$. So the above Lüscher's formulae reduce to their counterparts in the case of the meson-meson multichannel scattering [44].

We are now in a position to compare the formulae obtained using quantum field theory here and those in non-relativistic quantum mechanics in the previous section. Recall that (see the comments at the end of previous section), in the latter case, Lüscher's formula is expressed in terms of the function $\mathcal{M}_{i;J M l; J' M' l'}^{\mathbf{d}(s)}(k_i^{*2})$ while in the quantum field theory case it is expressed in terms of $F_{i;J M l; J' M' l'}^{FV(s)}$. It turns out that these two quantities are related by:

$$F_{i;J M l; J' M' l'}^{FV(s)} = i^l \mathcal{M}_{i;J M l; J' M' l'}^{\mathbf{d}(s)} i^{-l'} \quad (37)$$

as discussed in the appendix. This relation is valid up to terms that are suppressed exponentially by the box size. With this relation in mind, it becomes clear then that these two versions of Lüscher's formula are equivalent up to terms that are exponentially suppressed in the box size. As mentioned in the introduction, Briceno recently performed the most general study of two-particle system with arbitrary spin in a finite volume [51]. We have compared our results with his and agreements are found when comparable.

IV. CONCLUSIONS

In this paper, based on our previous works [1], we continue to discuss multi-channel Lüscher's formula for two-particle system with spin $\frac{1}{2}$. We have done this in

the cases of both non-relativistic quantum mechanics and quantum field theory. Finally, we show that the two versions of Lüscher's formula obtained within two different methods are equivalent up to terms that are exponentially suppressed in the box size. Our formula can be readily utilized in the study of baryon-baryon scattering, especially in the case of multi-channel scattering. A typical example would be $N\Sigma - N\Lambda$ coupled channel scattering which is important for the study of dense nuclear matter [53].

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Appendix A: Loop integration for the single channel

This appendix serves to fill the gap that leads to Eq. (37) which is utilized in the text to show the equivalence of Lüscher's formula obtained in non-relativistic quantum mechanics model and in quantum field theory. To setup the relation (37), we need to analyze intermediate loop integration/summation that appears in the Bethe-Salpeter equation. We have adopted the same no-

tation as in Ref. [1] and the reader is directed to that reference for unexplained notations. The derivation here is also quite brief and the details closely follows those in Ref. [1].

First, we quote a useful formula for the summation from Refs. [1, 25]. Below, quantities with a * indicate the corresponding values in the center-of-mass (COM) frame. The summation formula we need is:

$$S(\mathbf{q}^*) \equiv \frac{1}{L^3} \sum_{\mathbf{k}} \frac{w_{\mathbf{k}}^*}{w_{\mathbf{k}}} \frac{f^*(\mathbf{k})}{q^2 - \mathbf{k}^{*2}} \quad (\text{A1})$$

$$= \mathcal{P} \int \frac{d^3 k^*}{(2\pi)^3} \frac{f^*(\mathbf{k}^*)}{q^{*2} - k^{*2}} + \sum_{l,m} f_{lm} C_{lm}^P(q^{*2}) \quad (\text{A2})$$

where

$$f^*(\mathbf{k}^*) = \sum_{l=0}^{\infty} \sum_{m=-l}^l f_{lm}^*(k^*) k^{*l} \sqrt{4\pi} Y_{lm}(\hat{\mathbf{k}}^*) \quad (\text{A3})$$

and

$$C_{lm}^P(q^{*2}) = \frac{1}{L^3} \sum_{\mathbf{k}} \frac{w_{\mathbf{k}}^*}{w_{\mathbf{k}}} \frac{e^{\alpha(q^{*2} - k^{*2})}}{q^{*2} - k^{*2}} k^{*l} \sqrt{4\pi} Y_{lm}(\hat{\mathbf{k}}^*) - \mathcal{P} \int \frac{e^{\alpha(q^{*2} - k^{*2})}}{q^{*2} - k^{*2}} \quad (\text{A4})$$

where \mathcal{P} stands for the principal-value prescription. The above formulae will be utilized when we transform the intermediate loop integral/summation in the following.

Following similar steps as in Ref. [1], we need to calculate a loop integration/summation in the correlation function as

$$I = \frac{-1}{L^3} \sum_{\mathbf{k}} \int \frac{dk_0}{2\pi} \frac{f(k_0, \mathbf{k}) (k^\mu \gamma_\mu + m_1) \otimes ((P^\nu - k^\nu) \gamma_\nu + m_2)}{(k^2 - m_1^2 + i\epsilon)((P - k)^2 - m_2^2 + i\epsilon)} \quad (\text{A5})$$

In the process of this summation, we have to introduce a cutoff function $f(k_0, \mathbf{k})$ whose ultraviolet behavior renders the expressions convergent. The explicit form of this function is irrelevant but we will demand that $f(k_0, \mathbf{k})$ is an even function of \mathbf{k} , i.e. $f(k_0, -\mathbf{k}) = f(k_0, \mathbf{k})$. We can then divide the loop integration into two parts. The part that contains the two-particle poles in the finite volume is of interest here. Performing the k^0 integration the expression (A5) will pick up the relevant poles in the energy plane. Following similar steps as in Ref. [1], we obtain the function $C(\mathbf{k})$ that appeared in Eq. (25), which in

the COM frame becomes

$$C(\mathbf{k}^*) = [m_1 I \otimes E^* \gamma_0 - (k^{*\mu} \gamma_\mu) \otimes (k^{*\nu} \gamma_\nu) + m_1 m_2] |_{k^0 = w_{1\mathbf{k}}^*} \quad (\text{A6})$$

with $w_{1\mathbf{k}} = \sqrt{\mathbf{k}^2 + m_1^2}$. Using the summation formulae (A2), (A3), (A4) quoted at the beginning of this appendix, we find that the finite volume correction part of the loop integral (A5), I_{FV} , is given by

$$I_{FV} = \frac{q^* f_{00}^*(q^*) C(\mathbf{q}^*)}{8\pi E^*} - \frac{i C(\mathbf{q}^*)}{2E^*} \sum_{lm} f_{lm}^*(q^*) C_{lm}^P(q^{*2}). \quad (\text{A7})$$

Now we proceed as in Ref. [1]. By using the completeness of spherical harmonics, we are able to setup the following relations:

$$F \equiv C(\mathbf{q}^*) \tilde{F} = \frac{q^* C(\mathbf{q}^*)}{8\pi E^*} (1 + i F^{FV}), \quad (\text{A8})$$

$$\tilde{F}_{JMs;J'M'l's'} = \frac{q^*}{8\pi E^*} (\delta_{JJ'} \delta_{MM'} \delta_{ll'} \delta_{ss'} + i F_{JMs;J'M'l's'}^{FV}), \quad (\text{A9})$$

$$\begin{aligned} F_{JMs;J'M'l's'}^{FV} &= \frac{-4\pi}{q^*} \sum_{l_1 m_1} \frac{\sqrt{4\pi}}{q^{*l_1}} C_{l_1 m_1}^P \int d\Omega^* Y_{JM}^{ls\dagger} Y_{l_1 m_1}^* Y_{J'M'}^{l's'} \\ &= F_{JMs;J'M'l's'}^{FV(s)}, \end{aligned} \quad (\text{A10})$$

$$C_{l_1 m_1}^P(q^{*2}) = \frac{1}{L^3} \sum_{\mathbf{k}} \frac{w_{1\mathbf{k}}^*}{w_{1\mathbf{k}}} \frac{e^{\alpha(q^{*2}-k^{*2})}}{q^{*2}-k^{*2}} k^{*l_1} \sqrt{4\pi} Y_{l_1 m_1} - \mathcal{P} \int \frac{d^3 k^*}{(2\pi)^3} \frac{e^{\alpha(q^{*2}-k^{*2})}}{q^{*2}-k^{*2}}, \quad (\text{A11})$$

In Refs. [1, 54], for the scattering of two particles, the relations between the quantity $C_{lm}^P(q^{*2})$ and the function $Z_{lm}^{\mathbf{d}}(1, \kappa^2)$ were established. For the two channel case, this relation reads

$$C_{i;lm}^P(q_i^{*2}) = -\frac{\sqrt{4\pi}}{\gamma L^3} \left(\frac{2\pi}{L}\right)^{l-2} Z_{lm}^{\mathbf{d}}(1, \kappa_i^2) \quad (\text{A12})$$

Using this relation, we then arrive at the following equal-

ity:

$$F_{i;JMI;J'M'I'}^{FV(s)} = i^l \mathcal{M}_{i;JMI;J'M'I'}^{\mathbf{d}(s)} i^{-l'} \quad (\text{A13})$$

which is exactly Eq. (37) utilized in the text. These formulas hold up to terms that are vanishing exponentially in the box size.

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